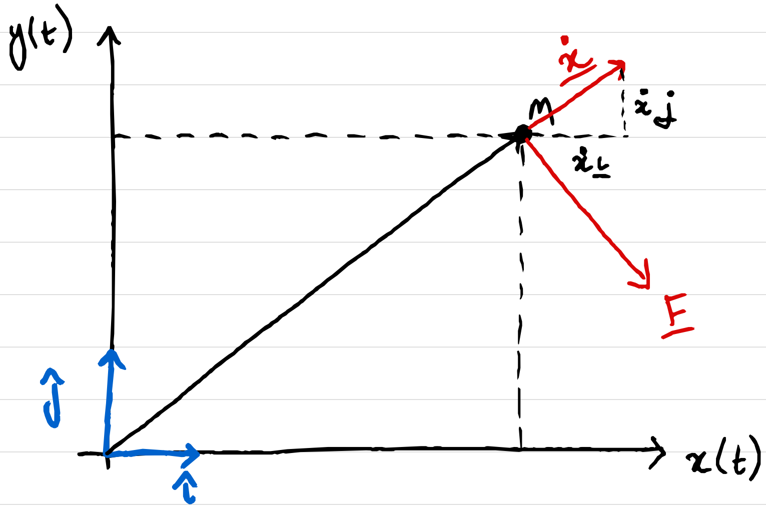
4) Solutions for Motion in 2D

4.1) Equations of motion in 2D

Aim is to consider 2 dimensional motion in a plane.

Consider a particle of mass m moving in a plane. To <u>quantitavely describe the position</u> of the particle, we <u>introduce Cartesian co-ordinates</u> in the plane as shown below.



<u>Defn:</u> <u>Position vector:</u>

The position of the particle can be described by its x and y co-ordinates, or equivalently by its position vector

$$\underline{x} = x\underline{i} + y\underline{j} = \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\underline{i}$ is the unit vector of x and $\underline{j}$ is the unit vector of y.

Defn: Force vector in 2D:

A force acting is also a vector.

$$\boxed{\vec{F} = F_x \hat{\imath} + F_y \hat{\jmath} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}}$$

Here x and y are both the x and y components of $\vec{F}$.

When particle is moving, its position vector was moving with time.

↳ That means both coordinates (components of position vector $\vec{x}$) are functions of time:

- $x = x(t)$
- $y = y(t)$

The position vector is also a vector valued function of time:

$$\boxed{\vec{x}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}}$$

The velocity $\vec{v}$ and acceleration $\vec{a}$ are defined as time derivatives of the position vector:

$$\boxed{\vec{v}(t) = \dot{\vec{x}}(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}}$$

$$\vec{a}(t) = \ddot{\vec{x}}(t) = \dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath} = \begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{pmatrix}$$

Now we can write:

Defn: Equation of motion in 2D

$$\boxed{m\ddot{\vec{x}} = \vec{F}}$$

or

$$\boxed{m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}F_x\\ F_y\end{pmatrix}}$$

Note that this vector equation can be treated as a system of 2 ODE's such that

$$\boxed{\begin{cases} m\ddot{x} = F_x \\ m\ddot{y} = F_y \end{cases}}$$

4.2 Motion under uniform gravity

Consider motion of a particle of mass m under uniform gravity force.

First we introduce Cartesian co-ordinates x, y such that

- x-axis is horizontal and represents the earths surface
- y-axis is vertical and directed upward

In this co-ordinate system, free fall acceleration vector is given by

$$\vec{g} = -g\,\mathbf{j}$$

The equation of motion of the particle is given by

$$m\ddot{\vec{x}} = m\vec{g} \Rightarrow \ddot{\vec{x}} = \vec{g}$$

$$\Rightarrow \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

So we have 2 simultaneous ODE's.

$$\ddot{x} = 0 \qquad (*1)$$

$$\ddot{y} = -g \qquad (*2)$$

The general solution of $(*1)$ is

$$x(t) = A_1 + B_1 t$$

The general solution of $(*2)$ is

$$y(t) = A_2 + B_2 t - gt^2/2$$

A_1, A_2, B_1, B_2 are arbitrary constants

We can also write the general solution of these 2 equations in a vector form as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + t\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} - \begin{pmatrix} 0 \\ gt^2/2 \end{pmatrix}$$

or

$$\vec{x} = \vec{A} + \vec{B} - \frac{gt^2}{2}\hat{j}$$

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

Note general solution contains 4 arbitrary constants A_1, A_2, B_1, B_2 or 4 arbitrary vectors $\vec{A}, \vec{B}$.

To select particular values, we impose <u>initial conditions</u>.

$$\vec{x}(0) = \vec{x_0} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

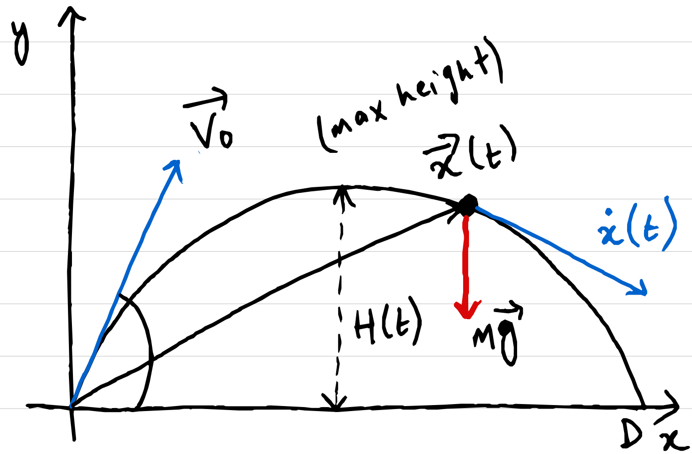
$$\vec{\dot{x}}(0) = \vec{v_0} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

Substituting into general solution we get

$$\vec{A} = \vec{x_0} \qquad \vec{B} = \vec{v_0}$$

So the solution that satisfies the initial conditions is

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0 t - \frac{gt^2}{2}\hat{j}$$



Example problem 1: (projectile motion):

A projectile is launched with initial speed $|\vec{v}_0|$

$$|\vec{v}_0| = V_0$$

at an angle θ to the horizontal.

(i) Find highest point of its trajectory

(ii) range D (distanced travelled in horizontal direction):

Solution:

As seen before, formula for solution satisfying initial condition

$$\vec{x}(0) = \vec{x}_0 = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\dot{\vec{x}}(0) = \vec{v}_0 = \begin{pmatrix} V_0 \cos\theta \\ V_0 \sin\theta \end{pmatrix}$$

Solution takes form

$$\vec{x}(t) = v_0 t - \frac{gt^2}{2} \hat{j}$$

or

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} V_0 \cos(\theta) t \\ V_0 \sin(\theta) t - gt^2/2 \end{pmatrix}$$

(i) To find highest point in trajectory, we need to find time t_1 that corresponds to this point.

Evidently, here, vertical velocity component is 0 at max height.

Therefore we obtain

$$\dot{y}(t_1) = 0 \Rightarrow V_0 \sin(\theta) - g t_1 = 0$$

$$\Rightarrow \boxed{t_1 = \frac{V_0 \sin\theta}{g}}$$

It follows that highest point of trajectory is

$$\vec{x}(t_1) = \begin{pmatrix} V_0 \cos(\theta) t_1 \\ V_0 \sin(\theta) t_1 - g t_1^2 \end{pmatrix}$$

$\Rightarrow$

$$\vec{x}(t_1) = \frac{V_0^2}{2g} \begin{pmatrix} \sin 2\theta \\ \sin^2\theta \end{pmatrix}$$

Now we know the co-ordinates of the highest point of the trajectory as a function of θ.

Evidently if $\theta = \frac{\pi}{2}$ (i.e. projectile is launched vertically) then

$$x(t_1) = 0$$

and maximum height is

$$y(t_1) = \frac{v_0^2}{2g}$$

which is the same answer we obtain in the 1D one.

(ii) Let t_2 be the time ball hits ground.

Here the vertical height is 0.

So $y(t_2) = v_0 \sin(\theta) t_2 - \frac{g t_2^2}{2} = 0$

$$\Rightarrow t_2 = \frac{2 v_0 \sin\theta}{g}$$

Hence range D is

$$D = x(t_2) = \frac{v_0^2}{g} \sin 2\theta.$$

The same answer can also be obtained by observing that the trajectory (which is sort of a parabola) is symmetric relative to its middle point so that

$$t_2 = 2t_1.$$

It follows that from the formula for D, maximum range is attained at $\theta = \frac{\pi}{4}$

range function:

$$D(\theta) = \frac{v_0^2}{g} \sin 2\theta$$

D_{max} when $D'(\theta) = 0$

$$D'(\theta) = 0 \Rightarrow \frac{d}{d\theta}\left(\frac{v_0^2}{g} \sin 2\theta\right)$$

$$\Rightarrow \frac{2v_0^2 \cos 2\theta}{g} = 0$$

$$\Rightarrow \cos 2\theta = 0 \Rightarrow 2\theta = \frac{\pi}{2}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{4}}$$

Remember: $\theta \in [0, \pi/2] \Rightarrow 2\theta \in [0, \pi]$

So

$$\boxed{D_{max} = \frac{v^2}{g}}$$

* Calculus Preliminary Overview:

- $\underline{\nabla} f(x,y)$ is called the gradient or grad function.

$$\underline{\nabla} : \mathbb{R} \to \mathbb{R}^2 \qquad \text{(in 2D)}$$

$$\underline{\nabla} f(x,y) = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \end{pmatrix}$$

- 2 vector valued functions $\vec{x}_1(t)$, $\vec{x}_2(t)$

$$\frac{d}{dt}\left[\vec{x}_1(t) . \vec{x}_2(t)\right] = \dot{\vec{x}}_1(t) . \vec{x}_2(t) + \dot{\vec{x}}_2(t)\, x_1(t)$$

(product rule)

- If $f : \mathbb{R}^2 \to \mathbb{R}$ and $x : \mathbb{R} \to \mathbb{R}^2$ then

$$f(\underline{x}(t)) : \mathbb{R} \to \mathbb{R}$$

satisfies

$$\frac{d}{dt} f(\vec{x}(t)) = \underbrace{(\underline{\nabla} f)(\vec{x}(t))}_{\text{grad function applied to } \vec{x}(t)} . \dot{\vec{x}}(t)$$

vector dot product

$$= \left[(\underline{\nabla} f)(x(t), y(t))\right] . \dot{\vec{x}}(t)$$

So $f(\vec{x}(t)) = f(x(t), y(t))$

$$\frac{d}{dt} f(x(t), y(t)) = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right)$$

$$= \frac{\partial f}{\partial t}\, \dot{x}(t) + \frac{\partial f}{\partial y}\, \dot{y}(t)$$

4.3) Motion in a potential

(under a conservative force)

- Consider a particle of mass m that is moving under a force depending on position of particle in 2 dimensions, i.e.

$$\boxed{\underline{F} = \underline{F}(\underline{x}) = F_x(x,y)\,\underline{i} + F_y(x,y)\,\underline{j} = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix}}$$

(considering canonical basis $\{\underline{i}, \underline{j}\}$)

Equations of motion take form:

$$m\ddot{\underline{x}} = \vec{F}(\vec{x}) \quad \text{or} \quad m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} F_x(x,y) \\ F_y(x,y) \end{pmatrix}$$

Now we will focus on a particular but very important form of forces: <u>conservative forces</u>.

4.3.1 Conservative forces:

- Given a potential $v(x,y)$

$$\boxed{\frac{d}{dt}v(x,y) = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} = (\nabla v).\dot{\underline{x}}}$$

- In 2D, kinetic energy is $T = \frac{1}{2}m\dot{\underline{x}}^2$

$$\boxed{T = \frac{1}{2}\dot{\underline{x}}^2 \Rightarrow T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)}$$

We expect energy to be in 2D

$$E = T + V(x,y)$$

$$\Rightarrow E = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + v(x,y)$$

* So F is a conservative force when E is a constant of motion, i.e.

$$\dot{E} = 0$$

$$\dot{E} = \frac{d}{dt}(E) = \frac{d}{dt}\left(\frac{m}{2}(\dot{x}^2+\dot{y}^2) + v(x,y)\right)$$

$$= \frac{m}{2}(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y}) + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= F_x\dot{x} + F_y\dot{y} + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y}$$

$$= \left(F_x + \frac{\partial V}{\partial x}\right)\dot{x} + \left(F_y + \frac{\partial V}{\partial y}\right)\dot{y}$$

$\Rightarrow$

$$\dot{E} = \left(F_x + \frac{\partial V}{\partial x}\right)\dot{x} + \left(F_y + \frac{\partial v}{\partial y}\right)\dot{y}$$

So

$$E = 0 \Rightarrow \left(F_x + \frac{\partial V}{\partial x}\right)\dot{x} + \left(F_y + \frac{\partial v}{\partial y}\right)\dot{y}$$

$$\Rightarrow F_x + \frac{\partial V}{\partial x} = 0 \text{ and } F_y + \frac{\partial V}{\partial y} = 0$$

$$\Rightarrow \boxed{F_x = -\frac{\partial V}{\partial x}} \text{ and } \boxed{F_y = -\frac{\partial V}{\partial y}}$$

Therefore

$$\boxed{\underline{F} = -\underline{\nabla} V}$$

Note: Here $\dot{\underline{x}}^2 = \dot{\underline{x}}.\dot{\underline{x}} = (x\underline{i} + y\underline{j}).(x\underline{i}+y\underline{j})$

↳ dot product

$$= x^2+y^2$$

$$\Rightarrow \underline{x}^2 = (x^2+y^2)$$

Also note:

$$T = \frac{1}{2}m\dot{\underline{x}}^2 \Rightarrow T = \frac{1}{2}m\dot{\underline{x}}.\dot{\underline{x}}$$

product rule $\rightarrow$ $\Rightarrow \frac{dT}{dt} = \frac{1}{2}m(\ddot{\underline{x}}.\dot{\underline{x}} + \dot{\underline{x}}.\ddot{\underline{x}})$

$$= \underline{F}.\dot{\underline{x}} \quad \text{(dot product)}$$

So $\frac{dT}{dt}$ = rate of change of energy

= power

$$\boxed{\text{Power} = \frac{dT}{dt} = \underline{F}.\dot{\underline{x}}}$$

Defn: Conservative force in 2D

In 2D $\underline{F}$ is conservative if there exists a potential function $V(\underline{x})$ (called the __potential__ of the force $\underline{F}$) such that

$$\boxed{\underline{F} = -\underline{\nabla} V = -\left(\frac{\partial V(x,y)}{\partial x}\underline{i}\,,\ \frac{\partial V(x,y)}{\partial y}\underline{j}\right) = -\begin{pmatrix} \partial V/\partial x \\ \partial V/\partial y \end{pmatrix}}$$

Note: If potential $V(\underline{x})$ exists, it is __not unique__. it is defined upto a constant.

If $\underline{F} = -\underline{\nabla} V(\underline{x})$ then the following is also true:

$$\underline{F} = -\underline{\nabla} V(\underline{x}) + c$$

So if $V(\underline{x})$ is a potential for a force, so is $V(\underline{x}) + c$

Only potentials that differ by constant give rise to the same force

- In contrast with one dimensional case, not all forces are conservative in 2D or 3D.

For example: Let

$$\underline{F}(\underline{x}) = \begin{pmatrix} \lambda y \\ -\lambda x \end{pmatrix}, \quad F_x = \lambda y, \quad F_y = -\lambda x$$

for a conservative force we would have relations

$$\lambda y = \frac{-\partial V(x,y)}{\partial x} \quad \text{and} \quad -\lambda x = -\frac{\partial V(x,y)}{\partial x}$$

or

$$\frac{\partial V(x,y)}{\partial x} = -\lambda y \qquad (*1)$$

$$\frac{\partial V(x,y)}{\partial x} = \lambda x \qquad (*2)$$

So solving $(*1)$ by indefinite integral

$$\frac{\partial V(x,y)}{\partial x} = -\lambda y \Rightarrow V = \int -\lambda y \, \partial x$$

$$\Rightarrow V = -\lambda xy + g(y)$$

A function of y. Note that in partial derivatives, integration in 1 variable yeilds not a constant but a <u>function</u> of the <u>other</u> variable $g(y)$

Substituting into $(*2)$ we get

$$\frac{\partial V(x,y)}{\partial x} = \lambda x \Rightarrow \frac{\partial}{\partial x}\left(-\lambda xy + g(y)\right) = \lambda x$$

$$\Rightarrow -\lambda x + g'(y) = \lambda x$$

$$\Rightarrow g'(y) = \lambda x$$

It is impossible to satisfy this equation for any choice of $g(y)$. So therefore no consist solution for V hence

$\underline{F}$ is not conservative.

Test for existence/non-existence of $V(x)$

There is a simple test to determine if the force is conservative or not.

Note: The test is based on the fact that if function $f(x,y)$ is sufficiently "good" (more precisely if all its second order partial derivatives are continuous) then

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \qquad (P)$$

Now lets assume that

$$\underline{F}(\underline{x}) = F_x(x,y)\hat{\imath} + F_y(x,y)\hat{\jmath}$$

is conservative.

Then there is $V(x,y)$ such that

$$\frac{\partial V}{\partial x} = -F_x(x,y) \qquad (*1)$$

$$\frac{\partial V}{\partial y} = -F_y(x,y) \qquad (*2)$$

Differentiating $(*1)$ with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = -\frac{\partial F_y}{\partial x} \qquad (*3)$$

Differentiating $(*2)$ with respect to x.

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = -\frac{\partial F_x}{\partial y} \qquad (*4)$$

Employing property (P) for $(*3)$ and $(*4)$ (Assuming that V is sufficiently smooth). This yeilds

$$\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right)$$

$\Rightarrow$

$$\frac{\partial F_x}{\partial x} = \frac{\partial F_y}{\partial y} \qquad (*5)$$

So if $(*5)$ is <u>not</u> satisfied then the force is <u>not conservative</u>.

The converse statement saying that if $(*5)$ is satisfied then the force is conservative is true under some restrictions

Propn: Conservative forces (Poincare's Lemma):

In 2D Euclidean plane, a force

$$\underline{F} = F_x(x,y)\underline{i} + F_y(x,y)\underline{j}$$

is conservative if and only if

$$\boxed{\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}}$$

4.3.2 Examples of Conservative forces

- uniform gravity:

If the y-axis is vertical and directed upward and x axis is parallel to the ground then

$$\underline{F} = m\underline{g} = -mg\underline{j} \quad \text{and} \quad V(\underline{x}) = V(x,y) = mgy$$

Lets verify that $V = mgy$ is the potential.

We have

$$\frac{\partial V}{\partial x} = 0, \quad \frac{\partial V}{\partial y} = -mg \Rightarrow \underline{F} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

$$\Rightarrow \underline{F} = -mg\underline{j}$$

as required.

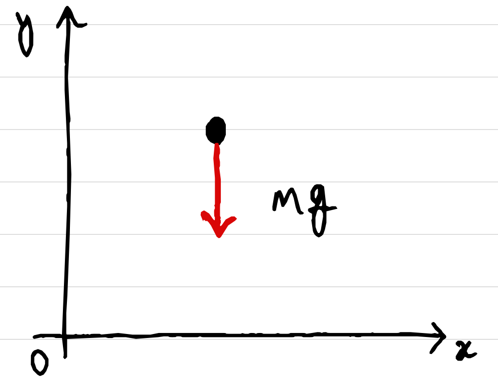
In fact any force of the form

$$\vec{F} = F_x(x)\hat{i} + F_y(y)\hat{j}$$

is conservative.

↳ since in 1D, individually $F_x(x)$ and $F_y(y)$ is conservative.

Diagram:



- Newtownian gravitation

The newtownian gravitational force of attraction between 2 bodies of mass m and M is

$$\frac{GMm}{r^2}$$

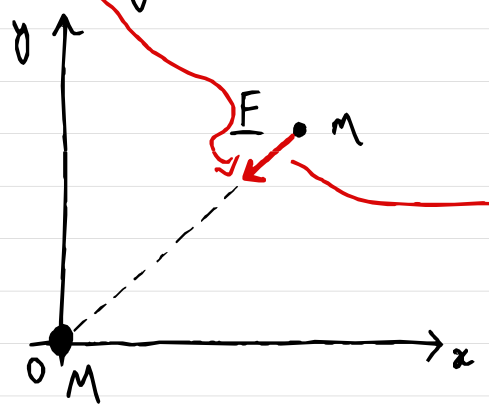
where G is the gravitational constant and r is the distance between the centre of the bodies.

Let M be the mass of the Earth.
Let m be the mass of another spherically symmetric body and let the Cartesian x and y axes be such that

(i) origin is at center of earth.

(ii) initial position of body lies in xy plane

(iii) inital velocity of body is parallel to xy plane.



The motion of body will be restricted to xy plane, and we have

$$\underline{F} = -\frac{GmM}{|\underline{x}|^2}\cdot\frac{\underline{x}}{|\underline{x}|}$$

- unit vector direction of force

says that force is opposite to direction, i.e. towards origin

So $\underline{F} = -\frac{GmM}{|\underline{x}|^2}\cdot\frac{\underline{x}}{|\underline{x}|}$ and $V(\underline{x}) = -\frac{GmM}{|\underline{x}|}$

where $|\underline{x}| = \sqrt{x^2+y^2}$

Showing that $V = -\frac{GmM}{|\underline{x}|}$ is the potential,

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{GMm}{|\underline{x}|}\right)$$

$$= -GMm\frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)^{1/2}}\right)$$

$$= +GMm\cdot\cancel{2}x\cdot(x^2+y^2)^{-3/2}\cdot\frac{1}{\cancel{2}}$$

$$= \frac{GMm}{|\underline{x}|^2}\cdot\frac{x}{|\underline{x}|}$$

$$\Rightarrow \frac{\partial V}{\partial x} = \frac{GMm}{|\underline{x}|^2}\cdot\frac{x}{|\underline{x}|}$$

Similarly

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-GMm}{|\underline{x}|}\right) = \frac{\partial}{\partial y}\left(\frac{-GMm}{(x^2+y^2)^{1/2}}\right)$$

$$= -GMm\frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2)^{1/2}}\right)$$

$$= +GMm\cdot\cancel{2}y\cdot(x^2+y^2)^{-3/2}\cdot\frac{1}{\cancel{2}}$$

$$= \frac{GMm}{|\underline{x}|^2}\cdot\frac{y}{|\underline{x}|}$$

$$\Rightarrow \frac{\partial V}{\partial y} = \frac{GMm}{|\underline{x}|^2}\cdot\frac{y}{|\underline{x}|}$$

Therefore

$$\underline{F} = \underline{\nabla} V$$

Showing that $\underline{F}$ is conservative

$$\underline{F} = \frac{-GMm}{|\underline{x}|^2} \cdot \frac{\underline{x}}{|\underline{x}|}$$

$$\Rightarrow \underline{F} = \frac{-GMm}{|\underline{x}|^{3/2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \underline{F} = \frac{-GMm}{(x^2+y^2)^{3/2}} \begin{pmatrix} x \\ y \end{pmatrix}$$

Therefore

$$F_x = \frac{-GmM}{(x^2+y^2)^{3/2}} x$$

$$F_y = \frac{-GmM}{(x^2+y^2)^{3/2}} y$$

$$\frac{\partial F_x}{\partial y} = -GMm \frac{\partial}{\partial y}\left(\frac{x}{(x^2+y^2)^{3/2}}\right)$$

$$= \cancel{-}GMm \cdot \cancel{2}y \cdot x \cdot \cancel{-}3/\cancel{2} \cdot (x^2+y^2)^{-5/2}$$

$$\Rightarrow \frac{\partial F_x}{\partial y} = \frac{3GMm\,xy}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial F_y}{\partial x} = -GMm \frac{\partial}{\partial x}\left(\frac{y}{(x^2+y^2)^{3/2}}\right)$$

$$= \cancel{-}GMm\,\cancel{2}xy\,(\cancel{-}3/\cancel{2})(x^2+y^2)^{-5/2}$$

$$\Rightarrow \frac{\partial F_y}{\partial x} = \frac{3GMm\,xy}{(x^2+y^2)^{5/2}}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$ $\Rightarrow$ force is conservative.

Example problem 2:

A particle moving in xy plane is under the action of force

$$\underline{F} = \begin{pmatrix} -Ax - Cy \\ -Cx - By \end{pmatrix} \qquad A, B, C > 0$$

Is this force conservative.
If yes, find potential:

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(-Cx - By\right) = -C$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(-Ax - Cy\right) = -C$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} \Rightarrow \text{force is conservative.}$$

Finding potential:

By definition of V:

$$\frac{\partial V(x,y)}{\partial x} = -F_x = -(-Ax - Cy) = Ax + Cy$$

and

$$\frac{\partial V(x,y)}{\partial y} = -F_y = -(-Cx - By) = Cx + By$$

Integrating the first

$$V = \int Ax + Cy \, \partial x$$

$$\Rightarrow V = \frac{1}{2}Ax^2 + Cyx + g(y)$$

where $g(y)$ is an arbitrary function of one variable

Finding $g(y)$:

Substituting $V = \frac{1}{2}Ax^2 + Cyx + g(y)$ into second eqn

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}Ax^2 + Cyx + g(y)\right) = Cx + By$$

$$\Rightarrow \cancel{Cx} + g'(y) = \cancel{Cx} + By$$

$$\Rightarrow g'(y) = By$$

$$\Rightarrow g(y) = \frac{By^2}{2} + D$$

where D is an arbitrary constant. So

$$V(x,y) = \frac{Ax^2}{2} + Cxy + \frac{By^2}{2} + D$$

4.4 Conservation of Energy

By analogy of one-dimensional motion, energy of particle moving in a potential $V(x,y)$ in 2 dimensions is given by

$$\boxed{E = \frac{m|\dot{\underline{x}}|^2}{2} + V(\underline{x})}$$

or

$$\boxed{E = \frac{m\dot{\underline{x}}^2}{2} + V(\underline{x})}$$

$\dot{\underline{x}}^2 = \dot{\underline{x}} . \dot{\underline{x}}$
(dot product)

or

$$\boxed{E = m\frac{\dot{x}^2 + \dot{y}^2}{2} + V(x,y)}$$

Here we use the notation $|\dot{\underline{x}}|^2 = \dot{\underline{x}} . \dot{\underline{x}} = \dot{x}^2 + \dot{y}^2$

Here

$T = \frac{m|\dot{\underline{x}}|^2}{2}$ is the kinetic energy

$V(x,y)$ is the potential energy

Question: Is the energy given by

$$\frac{m|\dot{\underline{x}}|^2}{2} + v(x,y)$$

a constant of motion:

For a particle moving in a potential $v(\underline{x})$, the equations of motion are

$$m\ddot{\underline{x}} = -\underline{\nabla} v \Rightarrow m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = -\begin{pmatrix}\partial v/\partial x\\ \partial v/\partial y\end{pmatrix}$$

Computing derivative of E with respect to t, we have

$$\frac{dE}{dt} = m(\dot{x}\ddot{x} + \dot{y}\ddot{y}) + \frac{\partial v}{\partial x}\dot{x} + \frac{\partial v}{\partial y}\dot{y}$$

$$= \dot{x}\left(m\ddot{x} + \frac{\partial v}{\partial x}\right) + \dot{y}\left(m\ddot{y} + \frac{\partial v}{\partial y}\right)$$

$$= \dot{x}\left(F_x + \frac{\partial v}{\partial x}\right) + \dot{y}\left(F_y + \frac{\partial v}{\partial y}\right)$$

$$= \dot{x}\left(\cancel{-\frac{\partial v}{\partial x}} + \cancel{\frac{\partial v}{\partial x}}\right) + \dot{y}\left(\cancel{-\frac{\partial v}{\partial y}} + \cancel{\frac{\partial v}{\partial y}}\right)$$

$$= 0 + 0$$

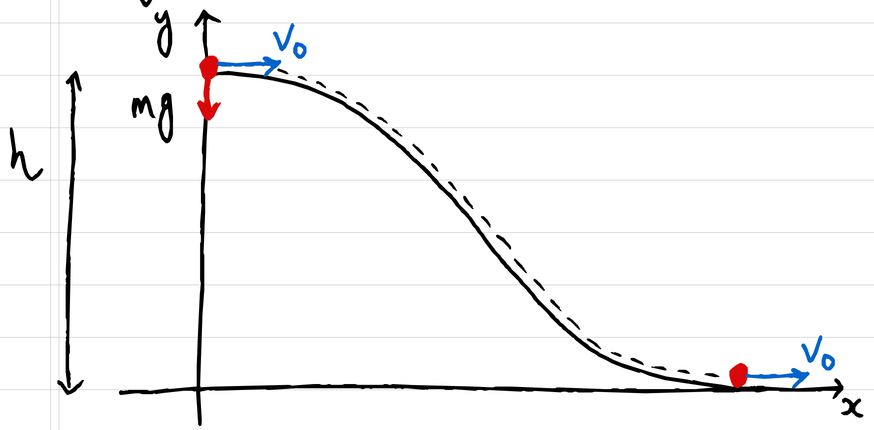
$$= 0$$

$\Rightarrow \dot{E} = 0 \Rightarrow$ energy is a constant of motion.

4.5) Solutions using energy conservation

Example problem 3: (Sliding down):

Consider a body of mass m that can slide along a smooth surface shown below:



There is no friction. Find velocity $\vec{v}_1 = (v_1, 0)$ of the body if its initial is $\vec{v}_0 = (v_0, 0)$

Soln:

The energy of body at time $t=0$ is

$$E(0) = \frac{m v_0^2}{2} + mgh$$

The energy of the body at time t_1 when its velocity is v_1 is given by

$$E(t_1) = \frac{m v_1^2}{2}$$

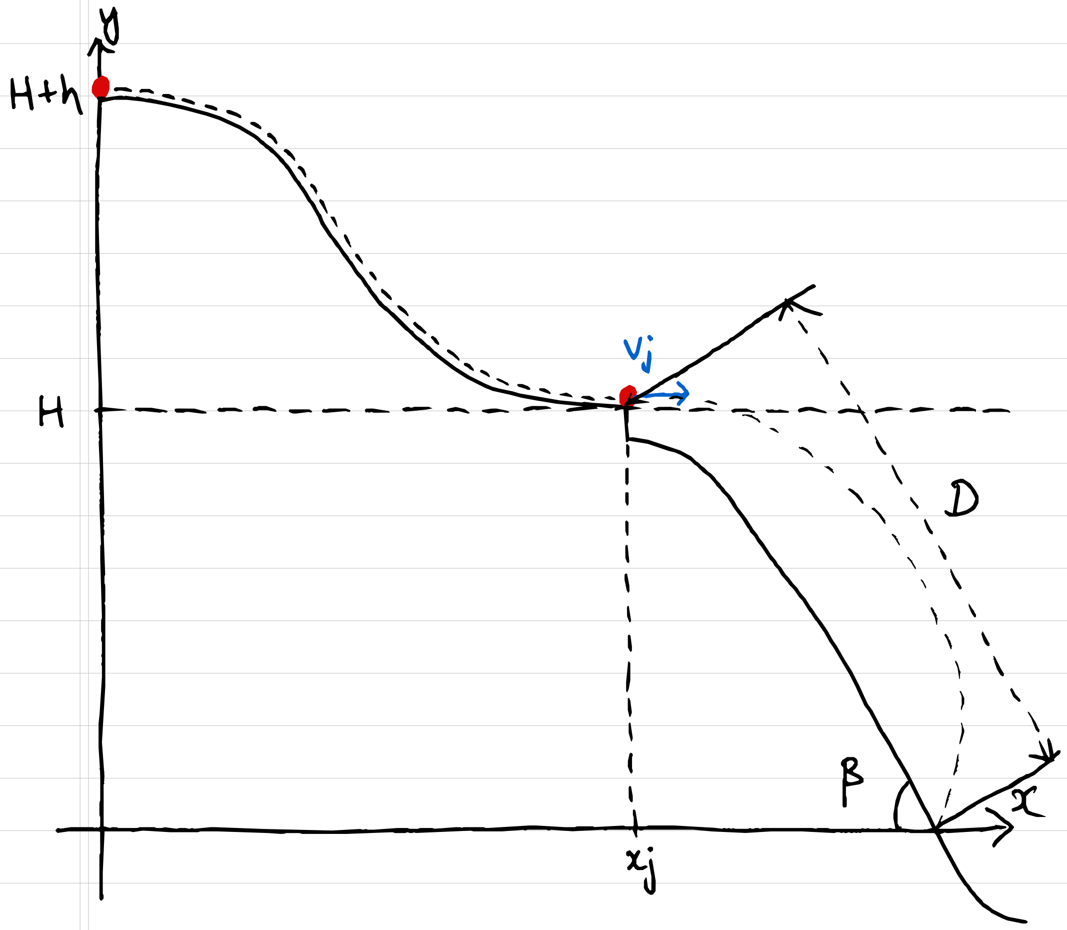
Since energy is a constant of motion, we obtain

$$E(0) = E(t_1) \Rightarrow \frac{m v_0^2}{2} + mgh = \frac{m v_1^2}{2}$$

$$\Rightarrow v_1 = \sqrt{v_0^2 + 2gh}$$

Example problem 4: (ski flying/skijumping):

Now let us try to estimate the length of the world record ski-jump. sketch is shown below:



Our aim is to find length of jump D.

To do this, we need to know the velocity at moment of take off,

$$\vec{v}_j = (v_j, 0)$$

and angle β.

We take $\beta = 35^\circ$

To find v_j, we need the difference in height between the starting point and take off

We take $h \approx 73m$

To determine v_j, employ conservation of energy.

$$E(0) = mg(H+h)$$

At take of time t_j,

$$E(t_j) = \frac{1}{2}mv_j^2 + mgH$$

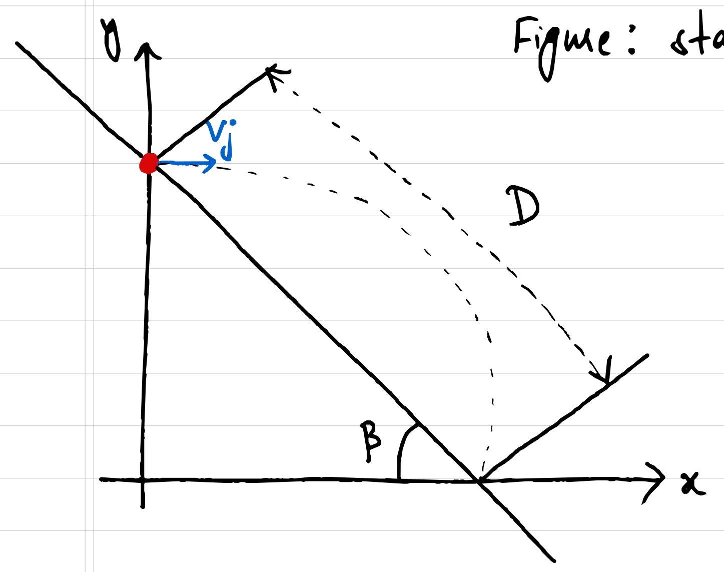
$$E(0) = E(t_j) \Rightarrow mg(H+h) = \frac{1}{2}mv_j^2 + mgH$$

$$\Rightarrow mgh = \frac{mv_j^2}{2}$$

$$\Rightarrow \boxed{v_j = \sqrt{2gh}}$$

$$\Rightarrow v_j \approx 38 m/s$$

Figure: stage of flight.



The stage of flight can be described by projectile motion.

The stage of flight can be described as the motion of the projectile above the inclined plane as shown above.

Let H be the height of the point of take off, above the point of landing.

Initially at time $t=0$, horizontal velocity

$$\vec{v_j} = (v_j, 0)$$

and position $\vec{x_j} = (0, H)$

The equations of motion are

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}0\\ -mg\end{pmatrix}$$

Solving these subject to initial conditions

$$x(0)=0, \quad y(0)=H$$

$$\dot{x}(0)=v_j \quad \dot{y}(0)=0$$

we obtain

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}v_j t\\ H - gt^2/2\end{pmatrix}$$

At time t_2, (moment of landing):

$$y(t_2) = 0$$

Hence

$$H - \frac{gt^2}{2} = 0 \Rightarrow t_2 = \sqrt{\frac{H}{g}}$$

It follows that x-coordinate of the point of landing is

$$x(t_2) = v_j t_2 = v_j\sqrt{\frac{2H}{g}}$$

On the other hand,

$$\tan(\beta) = \frac{H}{x(t_2)}$$

Therefore

$$\tan^2(\beta) = \frac{H^2}{(x(t_2))^2} = \frac{gH}{2v_j^2}$$

or

$$\boxed{H = \frac{2v_j^2}{g}\tan^2(\beta)}$$

Finally length of D is

$$D = \frac{H}{\sin(\beta)} \approx 253\,m$$

Example problem 5: (Taking off)

Consider particle of mass m sliding along a given smooth surface,

$$y = h(x)$$

without friction.

If its velocity is high enough, it sometimes can take of from surface and fly; under what conditions does it fly.

Solution:

To answer this question, first make an observation that when particle is moving on the surface, its velocity is tangent to it.

↳ i.e. parallel to the tangent line to the surface at the same point.

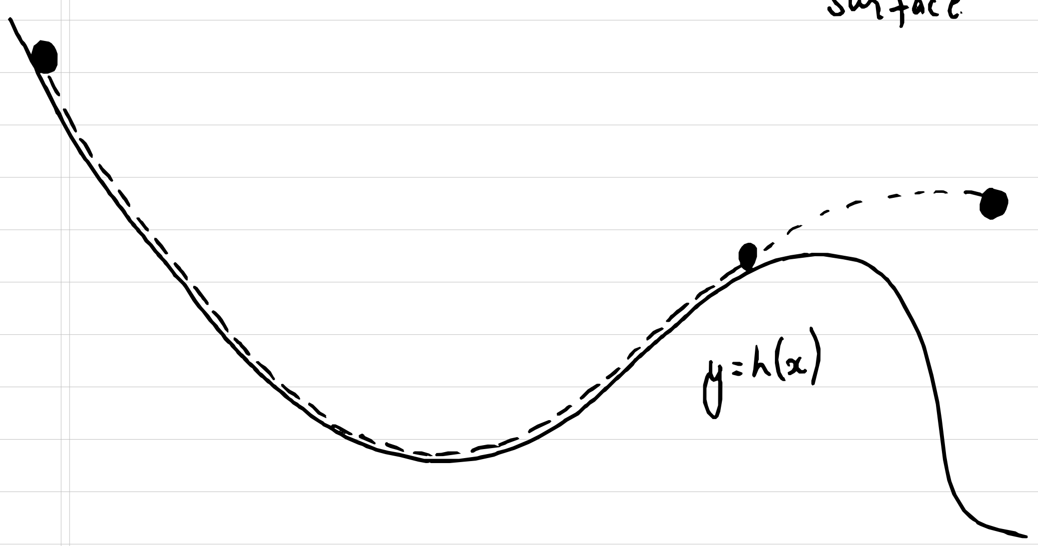
Let at time $t=0$, the particle at point

$$\underline{x}(0) = \underline{x}_0 = (x_0, y_0)$$

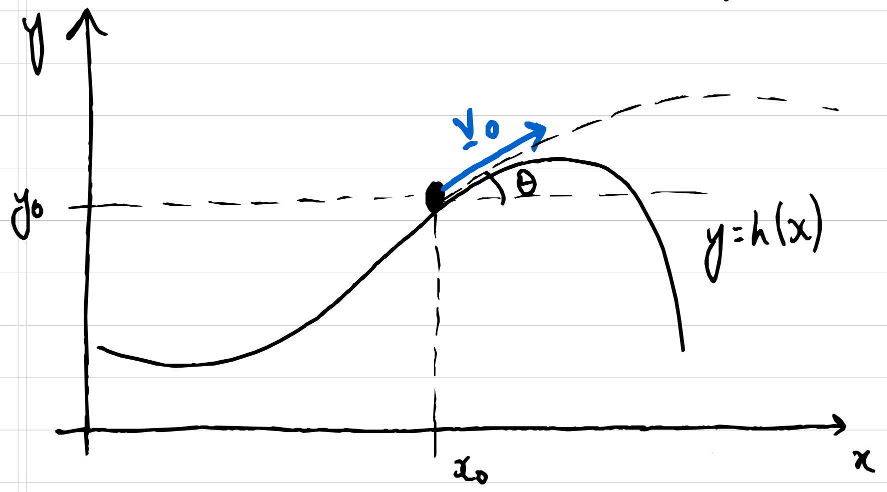
on the surface.

$$y_0 = h(x_0) \quad \text{where } y = h(x)$$

↳ equation of surface



Its velocity $\underline{v}_0$ is parallel to the tangent line to the surface at this point, which means that the angle θ satisfies the following relation.



$$\boxed{\tan\theta = f'(x_0)}$$

If $\underline{x_0}$ is the point of take-off, then the particle will fly for $t>0$ and its motion will be governed by equations of motion of a projectile:

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}0\\ -mg\end{pmatrix}$$

Solving these subject to initial conditions

$$x(0) = x_0 \qquad \dot{x}(0) = v_0\cos\theta$$

$$y(0) = y_0 \qquad \dot{y}(0) = v_0\sin\theta$$

we obtain

$$x(t) = x_0 + v_0\cos(\theta)\,t \qquad (*1)$$

$$y(t) = y_0 + v_0\sin(\theta)\,t - \frac{gt^2}{2} \qquad (*2)$$

Functions $x(t)$ and $y(t)$ represent parametric equation of particle trajectory.

It is convenient to rewrite it as

$$y = G(x)$$

(cartesian eqn of trajectory)

To do this, we eliminate t from eqn $(*1)$ and $(*2)$

$$\boxed{y = G(x) = y_0 + \tan(\theta)(x-x_0) - \frac{g}{v_0^2\cos^2\theta}\frac{(x-x_0)^2}{2}}$$

So as expected, we obtained eqn of parabola.

The particles trajectory

$$y = G(x)$$

will be above surface $y = h(x)$

$$\Delta y(x) = G(x) - h(x) > 0 \quad \text{for} \quad x > x_0$$

If we restrict our analysis to a small neighbourhood of x_0, we can expand $h(x)$ in Taylor Series about x_0, i.e.

$$h(x) = h(x_0) + (x - x_0)\, h'(x_0) + \frac{(x-x_0)^2}{2} h'(x_0) + \cdots$$

Substituting this into $\Delta y(x)$ and ignoring the higher order terms, we obtain

$$\Delta y(x) = y_0 - h(x_0) + (\tan(\theta) - h'(x_0)](x - x_0)$$
$$+ \left[\frac{-g}{v^2 \cos^2\theta} - h''(x_0)\right] \frac{(x-x_0)^2}{2}$$

Using the fact that $h'(x_0) = \tan\theta$ and $y_0 = h(x_0)$

$$\Delta y(x) = \left[\frac{-g}{v^2\cos^2\theta} - h''(x_0)\right]\frac{(x-x_0)^2}{2}$$

$\left(y(x(0)) = y(x_0) = h(x_0)\right)$

So $\Delta y(x) > 0$ provided

$$-\frac{g}{v^2 \cos^2\theta} - h''(x_0) > 0$$

$\iff$

$$h''(x_0) < \frac{-g}{v_0^2 \cos^2(\theta)}$$

We eliminate $\cos^2\theta$ with help of identity
$1 + \tan^2\theta = 1/\cos^2\theta = 1 + (h'(x_0))^2$

$$\frac{h''(x_0)}{1 + (h'(x_0))^2} > \frac{g}{V_0^2}$$

This inequality means surface is must be concave (negative $h''(x_0)$)

Also implies that V_0 cannot be too small (for any given $h(x)$ such that $h''(x_0) < 0$), the above inequality will not be satisfied for sufficiently small v_0.

Example problem 6: (A conservative forces):

Consider a particle of mass m moving under the action of a conservative force

$$\underline{F} = -Ax\underline{i} - By\underline{j}$$

where A and B are positive constants.

(a) Find the potential $V(\underline{x})$

(b) Write down equations of motion, and solve them subject to initial conditions

$$\underline{x}(0) = x_0\underline{i} + y_0\underline{j}$$

$$\dot{\underline{x}}(0) = u_0\underline{i} + y_0\underline{j}$$

(c) Find a condition on constants A and B which must be satisfied for the trajectory of the particle to be the closed curve on the (x,y) plane.

Solution (a):

If $v(\underline{x})$ is a potential, then

$$\frac{\partial V}{\partial x} = -F_x \qquad \Longrightarrow \qquad \frac{\partial V}{\partial x} = Ax$$

$$\frac{\partial V}{\partial y} = -F_y \qquad\qquad \frac{\partial V}{\partial y} = By$$

Integrating first equation in x,

$$v(x,y) = \frac{Ax^2}{2} + g(y) \qquad \left(\frac{\partial v}{\partial y} = 0 + g'(y)\right)$$

Substituting into second:

$$\frac{\partial V}{\partial y} = By = g'(y) \Rightarrow g(y) = \frac{By^2}{2} + D$$

Choose $D=0$. So

$$\boxed{v(x,y) = \frac{Ax^2 + By^2}{2}}$$

(b)

Equation of motion:

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}-Ax\\ -By\end{pmatrix} \quad \text{or}$$

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}-w_1^2 x\\ -w_1^2 y\end{pmatrix}$$

where $w_1^2 = \frac{A}{m}$, $w_2^2 = \frac{B}{m}$

Thus we have the following system of two scalar equations

$$\begin{cases}\ddot{x} + w_1 x = 0 & (*1)\\ \ddot{y} + w_2 y = 0 & (*2)\end{cases}$$

These solved subject to initial conditions

$$x(0) = x_0 \quad \dot{x}(0) = u_0$$

$$y(0) = y_0 \quad \dot{y}(0) = v_0$$

Each equation coincides with eqn of simple harmonic motion oscillator, solved earlier

$$x(t) = x_0\cos(w_1 t) + \frac{u_0}{w_1}\sin(w_1 t)$$

$$y(t) = y_0\cos(w_2 t) + \frac{v_0}{w_2}\sin(w_2 t)$$

(c) To explain behaviour of trajectories, it is convenient to present solution in form

$$x(t) = A_1\sin(w_1 t + \delta_1)$$

$$y(t) = A_2\sin(w_2 t + \delta_2)$$

where $A_1, A_2 > 0$, $\delta_1, \delta_2 \in [0, 2\pi)$

Remark:

If function $f(t)$ is periodic with period T, i.e.

$$f(t+T) = f(t) \quad \forall t \in \mathbb{R}$$

then it is also periodic with periods $2T$, $3T$, etc.
i.e. periodic with period nt for $n \in \mathbb{N}$

Example: functions $\sin t$ and $\cos t$ are periodic with $T = 2\pi n$ for $n \in \mathbb{N}$

If we say that the trajectory of the particle is a closed curve in the (x,y) plane, this means that there is $T > 0$ such that

$$\underline{x}(t+T) = \underline{x}(t) \quad \text{or equivalently}$$

$$\begin{cases} x(t+T) = x(t) \\ y(t+T) = y(t) \end{cases} \qquad \text{for } t \in \mathbb{R}$$

It follows from solutions

$$x(t) = A_1 \sin(w_1 t + \delta_1)$$

$$y(t) = A_2 \sin(w_2 t + \delta_2)$$

The above condition is equivalent to

$$\sin(w_1(x+T)+\delta_1) = \sin(w_1 t + \delta_1 + 2\pi n_1)$$

$$\Rightarrow \quad w_1 T = 2\pi n_1$$

Similarly $w_2 T = 2\pi n_2$

for some $n_1, n_2 \in \mathbb{Z}$. These imply that

$$T = \frac{2\pi n_1}{w_1} = \frac{2\pi n_2}{w_2}$$

The last equality only possible if

$$\boxed{\frac{w_1}{w_2} = \frac{n_1}{n_2}}$$

This means that the trajectory of the particle will be a closed curve only if the ratio of frequencies a rational number

i.e.

$$\boxed{\frac{w_1}{w_2} \in \mathbb{Q} \quad \Rightarrow \quad \text{trajectory is closed}}$$

Defn: Commensurable or not:

Two frequencies with rational ratio is called commensurable.

$$\frac{\omega_1}{\omega_2} \in \mathbb{Q}$$

If the ratio of two frequencies is an irrational number, they are called incommensurable

$$\frac{\omega_1}{\omega_2} \in \mathbb{R} \setminus \mathbb{Q}$$